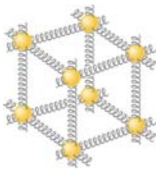
Introduction to

Static/Dynamic Deformation with Mass-Spring Systems



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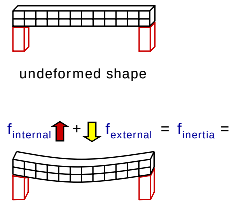
Static/Dynamic Deformation

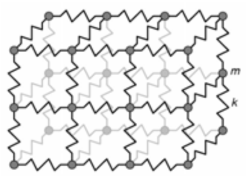
- Static deformation



- Dynamic deformation

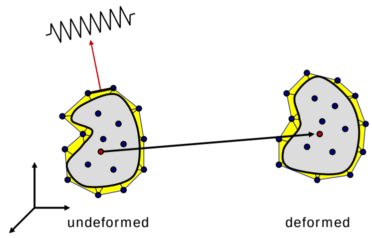
Mass-Spring Systems

- Using spring forces to connect mass points



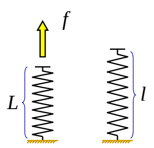
Mass-Spring Systems

- Using spring forces to connect mass points



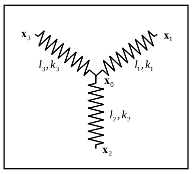
Elastic Springs

- **Spring characteristics**
  - Stiffness constant: k
  - Initial spring length: l
  - Current spring length: L
- **Hooke's law:** $f = k(L - l)$



Forces at the Mass Points

- **Internal force** $\mathbf{f}_i^{int}$
  - $\mathbf{f}_0^{int} = \sum_{i=1}^{3} k_i\left(\|\mathbf{x}_i - \mathbf{x}_0\| - l_i\right)\frac{\mathbf{x}_i - \mathbf{x}_0}{\|\mathbf{x}_i - \mathbf{x}_0\|}$
- **External force** $\mathbf{f}_i^{ext}$
  - Gravity, etc.
- **Resulting force** $\mathbf{f}_i = \mathbf{f}_i^{int} + \mathbf{f}_i^{ext}$



model problem

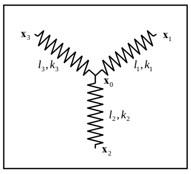
Static Equilibrium

- System of mass points

$$\mathbf{f}^{\text{int}} = \mathbf{K}(\mathbf{x})$$

- Static equilibrium

$$\mathbf{f} = \mathbf{K}(\mathbf{x}) + \mathbf{f}^{\text{ext}} = 0$$



model problem

From Static Eq. to Dynamic Eq.

- Static equilibrium

$$\mathbf{f} = \mathbf{K}(\mathbf{x}) + \mathbf{f}^{\text{ext}} = 0 \quad \Rightarrow \quad \mathbf{f}(t) = \mathbf{K}(\mathbf{x}(t)) + \mathbf{f}^{\text{ext}} = 0$$

- Equation of motion

$$\mathbf{M}\frac{d^2\mathbf{x}(t)}{dt^2} = \mathbf{f}(t)$$

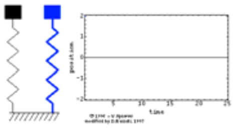
From Static Eq. to Dynamic Eq.

- Static equilibrium

$$\mathbf{f} = \mathbf{K}(\mathbf{x}) + \mathbf{f}^{\text{ext}} = 0 \quad \Rightarrow \quad \mathbf{f}(t) = \mathbf{K}(\mathbf{x}(t)) + \mathbf{f}^{\text{ext}} = 0$$

- Equation of motion with damping

$$\mathbf{M}\frac{d^2\mathbf{x}(t)}{dt^2} + \mathbf{C}\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(t)$$



Mass-Spring Dynamics

- Equation of motion for mass point i at time t

mass, position, damping coefficient

$$m_i \frac{d^2\mathbf{x}_i(t)}{dt^2} + c_i \frac{d\mathbf{x}_i(t)}{dt} = \mathbf{f}_i(t)$$

acceleration, damping, spring + external

- 2nd order differential equation
- f is used for acceleration and damping
- Without damping term, simply $f = ma$

Numerical Integration

- 2nd order ODE ⇒ two coupled 1st order ODEs

$$m_i \frac{d^2\mathbf{x}_i(t)}{dt^2} + c_i \frac{d\mathbf{x}_i(t)}{dt} = \mathbf{f}_i(t)$$

⇓

$$\begin{cases} \dfrac{d\mathbf{x}_i(t)}{dt} = \mathbf{v}_i(t) & \text{velocity} \\ \dfrac{d\mathbf{v}_i(t)}{dt} = \dfrac{\mathbf{f}_i(t) - c_i\mathbf{v}_i(t)}{m_i} & \text{acceleration} \end{cases}$$

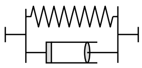
Damping Revisited

- Point damping

$$\frac{d\mathbf{v}_i(t)}{dt} = \frac{\mathbf{f}_i(t) - c_i\mathbf{v}_i(t)}{m_i}$$

- Damping force in opposite direction to the velocity

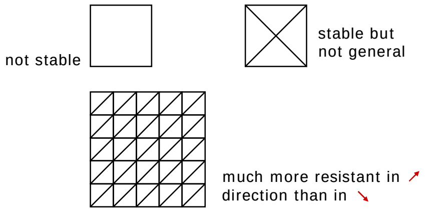
- Damped spring



- Damping force proportional to the velocity of the spring

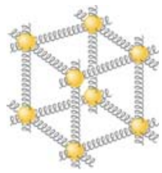
Topology and Stability

- Stability with respect to deformation



Requires well-chosen stable topologies!

Finite Element Method with Lumped Mass Formulation



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Mass-Spring Systems vs. FEM

- **Mass-spring dynamics**

$$m_i \frac{d^2\mathbf{x}_i(t)}{dt^2} + c_i \frac{d\mathbf{x}_i(t)}{dt} = \mathbf{f}_i(t)$$

- **Explicit integration**

$$\Delta\mathbf{x}_i = h(\dot{\mathbf{x}}_i + \Delta\dot{\mathbf{x}}_i)$$
$$\Delta\dot{\mathbf{x}}_i = \frac{h}{m_i}(\mathbf{f}_i - c_i\dot{\mathbf{x}}_i)$$

stable topology
appropriate constants

- **Elasto-dynamic equations**

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}$$

- **Explicit integration**

$$\Delta\mathbf{u} = h(\dot{\mathbf{u}} + \Delta\dot{\mathbf{u}})$$
$$\mathbf{M}\Delta\dot{\mathbf{u}} = h(\mathbf{f} - \mathbf{C}\dot{\mathbf{u}} - \mathbf{K}\mathbf{u})$$

matrix inversion
small displacements

Mass-Tensor Systems

- **Mass-spring dynamics**

$$m_i \frac{d^2\mathbf{x}_i(t)}{dt^2} + c_i \frac{d\mathbf{x}_i(t)}{dt} = \mathbf{f}_i(t)$$

- **Explicit integration**

$$\Delta\mathbf{x}_i = h(\dot{\mathbf{x}}_i + \Delta\dot{\mathbf{x}}_i)$$
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- **Elasto-dynamic equations**

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}$$

- **Explicit integration**

$$\Delta\mathbf{u} = h(\dot{\mathbf{u}} + \Delta\dot{\mathbf{u}})$$
$$\mathbf{M}\Delta\dot{\mathbf{u}} = h(\mathbf{f} - \mathbf{C}\dot{\mathbf{u}} - \mathbf{K}\mathbf{u})$$

Providing force from strain energy rather than spring energy!

Lumped Mass Formulation

- **Mass-spring dynamics**

$$m_i \frac{d^2\mathbf{x}_i(t)}{dt^2} + c_i \frac{d\mathbf{x}_i(t)}{dt} = \mathbf{f}_i(t)$$

- **Explicit integration**

$$\Delta\mathbf{x}_i = h(\dot{\mathbf{x}}_i + \Delta\dot{\mathbf{x}}_i)$$
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- **Elasto-dynamic equations**

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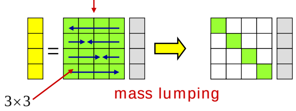


Lumped Mass Formulation

- **Mass-spring dynamics**

$$m_i \frac{d^2\mathbf{x}_i(t)}{dt^2} + c_i \frac{d\mathbf{x}_i(t)}{dt} = \mathbf{f}_i(t)$$

- **Explicit integration**

$$\Delta\mathbf{x}_i = h(\dot{\mathbf{x}}_i + \Delta\dot{\mathbf{x}}_i)$$
$$\Delta\dot{\mathbf{x}}_i = \frac{h}{m_i}(\mathbf{f}_i - c_i\dot{\mathbf{x}}_i)$$

- **Elasto-dynamic equations**

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}$$

- **Explicit integration**

$$\Delta\mathbf{u} = h(\dot{\mathbf{u}} + \Delta\dot{\mathbf{u}})$$
$$\mathbf{M}\Delta\dot{\mathbf{u}} = h(\mathbf{f} - \mathbf{C}\dot{\mathbf{u}} - \mathbf{K}\mathbf{u})$$

Mass Lumping ⇩

$$\Delta\mathbf{u} = h(\dot{\mathbf{u}} + \Delta\dot{\mathbf{u}})$$
$$\mathbf{D}_m\Delta\dot{\mathbf{u}} = h(\mathbf{f} - \mathbf{C}\dot{\mathbf{u}} - \mathbf{K}\mathbf{u})$$

⇦

$$\Delta\mathbf{u} = h(\dot{\mathbf{u}} + \Delta\dot{\mathbf{u}})$$
$$\Delta\dot{\mathbf{u}} = h\mathbf{D}_m^{-1}(\mathbf{f} - \mathbf{C}\dot{\mathbf{u}} - \mathbf{K}\mathbf{u})$$

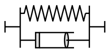
Mass-Spring vs. Lumped Mass FEM

- **Mass-spring dynamics**

$$m_i \frac{d^2\mathbf{x}_i(t)}{dt^2} + c_i \frac{d\mathbf{x}_i(t)}{dt} = \mathbf{f}_i(t)$$

Large displacements
are supported inherently.

Damped spring



- **Elasto-dynamic equations**

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}$$

Large displacements

$$\varepsilon_{ij} = \frac{\partial \mathbf{x}}{\partial u_i} \cdot \frac{\partial \mathbf{x}}{\partial u_j} - \delta_{ij}$$

Strain rate tensor

$$\nu_{ij} = \frac{\partial \mathbf{x}}{\partial u_i} \cdot \frac{\partial \dot{\mathbf{x}}}{\partial u_j} + \frac{\partial \dot{\mathbf{x}}}{\partial u_i} \cdot \frac{\partial \mathbf{x}}{\partial u_j}$$

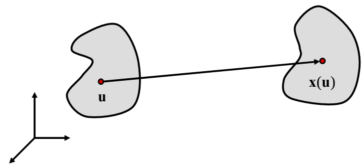
Adopt nonlinear visco-elastic formulation!

Visco-Elastic Formulation

- **Green's strain tensor**

$$\varepsilon_{ij} = \frac{\partial \mathbf{x}}{\partial u_i} \cdot \frac{\partial \mathbf{x}}{\partial u_j} - \delta_{ij} \quad \Rightarrow \quad \boldsymbol{\varepsilon} = \mathbf{J}^{\mathrm{T}}\mathbf{J} - \mathbf{I}$$

$$\mathbf{J} = \lfloor \partial x_i / \partial u_j \rfloor_{3\times 3}$$



Visco-Elastic Formulation

- **Green's strain tensor**

$$\varepsilon_{ij} = \frac{\partial \mathbf{x}}{\partial u_i} \cdot \frac{\partial \mathbf{x}}{\partial u_j} - \delta_{ij} \quad \Rightarrow \quad \boldsymbol{\varepsilon} = \mathbf{J}^{\mathrm{T}}\mathbf{J} - \mathbf{I}$$

$$\mathbf{J} = \lfloor \partial x_i / \partial u_j \rfloor_{3\times 3}$$

- **Strain rate tensor**

$$\dot{\mathbf{J}} = \lfloor \partial \dot{x}_i / \partial u_j \rfloor_{3\times 3}$$

$$\nu_{ij} = \frac{d\varepsilon_{ij}}{dt} = \frac{\partial \mathbf{x}}{\partial u_i} \cdot \frac{\partial \dot{\mathbf{x}}}{\partial u_j} + \frac{\partial \dot{\mathbf{x}}}{\partial u_i} \cdot \frac{\partial \mathbf{x}}{\partial u_j} \quad \Rightarrow \quad \mathbf{v} = \dot{\mathbf{J}}^{\mathrm{T}}\mathbf{J} + \mathbf{J}^{\mathrm{T}}\dot{\mathbf{J}}$$

Visco-Elastic Formulation

- **Stress tensor**

$$\boldsymbol{\sigma}^{\varepsilon} = \lambda\, \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon} \qquad \boldsymbol{\sigma}^{\nu} = \phi\, \mathrm{tr}(\mathbf{v})\mathbf{I} + 2\psi\mathbf{v}$$

elastic stress tensor — viscous stress tensor

- **Strain energy**

$$U^{\varepsilon} = \frac{1}{2}\int_{\Omega} \sigma_{ij}^{\varepsilon}\varepsilon_{ij}\, d\Omega \qquad U^{\nu} = \frac{1}{2}\int_{\Omega} \sigma_{ij}^{\nu}\nu_{ij}\, d\Omega$$

elastic potential energy — damping potential energy

Finite Element Discretization

- **Refer to the paper**

James F. O'Brien and Jessica K. Hogdins
Graphical Modeling and Animation of Brittle Fracture
Computer Graphics (*Proc. SIGGRAPH* '99), pp. 137-146