Introduction to

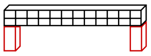
Static/Dynamic Deformation with Finite Element Method

Min Gyu Choi

Graphics & Media Lab.@SNU

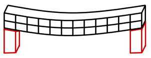
Static/Dynamic Deformation

- ☐ **Static deformation**



undeformed shape

$\mathrm{f}_{\mathrm{internal}}$ ⇧ + ⇩ $\mathrm{f}_{\mathrm{external}} = \mathrm{f}_{\mathrm{inertia}} =$



deformed shape

static equilibrium

- ☐ **Dynamic deformation**

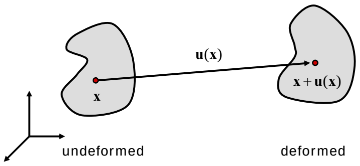
Static Deformation

- ☐ **Governing equations: Lagrange equations**

$$\frac{\partial U(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{f}_{ext}$$

elastic potential energy

external force



Static Deformation: Overview

- ☐ **Governing equations: Lagrange equations**

$$\frac{\partial U(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{f}_{ext}$$

elastic potential energy

external force

1. **Derive the elastic potential energy**
   - ☐ in terms of strain and stress

Review

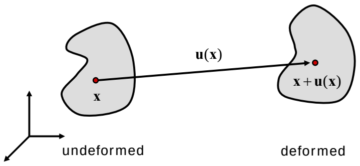
2. **Solve the partial differential equations**
   - ☐ with finite element discretization

Strain Tensor (review)

- ☐ **Small deformation linear strain tensor**

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

measures metric distortion



Strain Tensor (review)

- ☐ **Small deformation linear strain tensor**

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

measures metric distortion

- ■ 3x3 symmetric matrix representation

$$\mathbf{e} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

Strain Tensor (review)

- ☐ **Small deformation linear strain tensor**

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

measures metric distortion

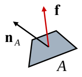
- ■ 6x1 column vector representation for convenience

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{bmatrix} = \begin{bmatrix} \partial/\partial x_1 & 0 & 0 \\ 0 & \partial/\partial x_2 & 0 \\ 0 & 0 & \partial/\partial x_3 \\ \partial/\partial x_2 & \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_3 & \partial/\partial x_2 \\ \partial/\partial x_3 & 0 & \partial/\partial x_1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Stress Tensor (review)

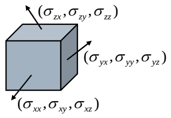
- ☐ **Stress is force per (oriented) area**

$$\mathbf{s} = \frac{d\mathbf{f}}{d\mathbf{A}} = \frac{d\mathbf{f}}{dA\cdot \mathbf{n}_A}$$



$$\frac{d\mathbf{f}}{dA} = \mathbf{s}\cdot\mathbf{n}_A = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \cdot \mathbf{n}_A$$

3x3 symmetric matrix



$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{yz} & \sigma_{zx} \end{bmatrix}^{\mathrm{T}}$$

6x1 column vector

Stress-Strain Relation (review)

- ☐ **Hooke's law:**

$$\mathbf{s} = \lambda\,\mathrm{tr}(\mathbf{e})\mathbf{I} + 2\mu\mathbf{e} \quad \Rightarrow \quad \boldsymbol{\sigma} = \mathbf{E}\cdot\boldsymbol{\varepsilon}$$

3x3 symmetric matrix — 6x1 column vector

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & 0 & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & \mu & 0 & 0 \\ & 0 & & 0 & \mu & 0 \\ & & & 0 & 0 & \mu \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{bmatrix}$$

Elastic Potential Energy (review)

- ☐ **Strain energy ⇐ "displacement x force"**

$$U = \frac{1}{2}\int_V \underbrace{e_{ij}s_{ij}}_{\text{strain x stress}}\,dV = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma}\,dV \overset{\boldsymbol{\sigma} = \mathbf{E}\cdot\boldsymbol{\varepsilon}\ \ Hooke's\ Law}{=} \frac{1}{2}\int_V \boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{E}\boldsymbol{\varepsilon}\,dV$$

strain x stress

$$\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{bmatrix} \cdot \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix}$$

$\boldsymbol{\sigma} = \mathbf{E}\cdot\boldsymbol{\varepsilon}$ Hooke's Law

End of Review

Static Deformation: Overview

- ☐ **Governing equations: Lagrange equations**

$$\frac{\partial U(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{f}_{ext}$$

elastic potential energy

external force

1. **Derive the elastic potential energy**
   - ☐ in terms of strain and stress — Review
2. **Solve the partial differential equations**
   - ☐ with finite element discretization

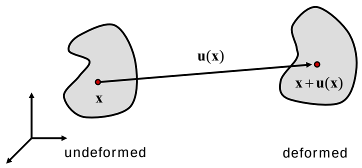
Finite Element Discretization

- ☐ **Governing equations: Lagrange equations**

$$\frac{\partial U(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{f}_{ext}$$

elastic potential energy

external force



undeformed — deformed

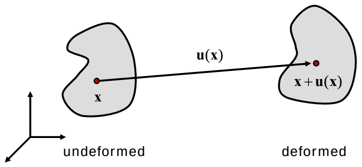
Finite Element Discretization

- ☐ **Governing equations: Lagrange equations**

$$U = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{E}\boldsymbol{\varepsilon}\,dV$$

$$\frac{\partial U(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{f}_{ext}$$

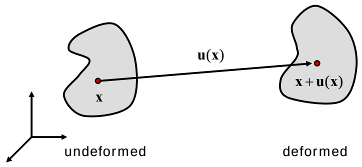
external force



Finite Element Discretization

- ☐ **Governing equations: Lagrange equations**

$$\frac{\partial U(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{f}_{ext} \Rightarrow \frac{\partial}{\partial \mathbf{u}}\left(\frac{1}{2}\int_V \boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{E}\boldsymbol{\varepsilon}\,dV\right) = \mathbf{f}_{ext}$$

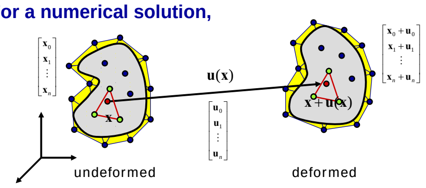


Finite Element Discretization

- ☐ **Governing equations: Lagrange equations**

$$\frac{\partial U(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{f}_{ext} \Rightarrow \frac{\partial}{\partial \mathbf{u}}\left(\frac{1}{2}\int_V \boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{E}\boldsymbol{\varepsilon}\,dV\right) = \mathbf{f}_{ext}$$

- ☐ **For a numerical solution,**

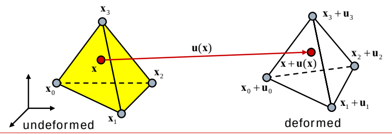


Linear Tetrahedral Element

- ☐ **Barycentric coordinate**

$$\begin{bmatrix}\mathbf{x}\\1\end{bmatrix} = \begin{bmatrix}\mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3\\1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}b_0\\b_1\\b_2\\b_3\end{bmatrix} = \mathbf{Xb}$$

$$\mathbf{b} = \mathbf{X}^{-1}\begin{bmatrix}\mathbf{x}\\1\end{bmatrix}$$

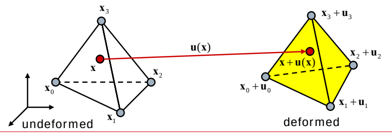


Linear Tetrahedral Element

- ☐ **Displacement function**

$$\begin{bmatrix}\mathbf{u}\\1\end{bmatrix} = \begin{bmatrix}\mathbf{u}_0 & \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3\\1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}b_0\\b_1\\b_2\\b_3\end{bmatrix}$$

$$\mathbf{b} = \mathbf{X}^{-1}\begin{bmatrix}\mathbf{x}\\1\end{bmatrix}$$



Linear Tetrahedral Element

- ☐ **Displacement function**

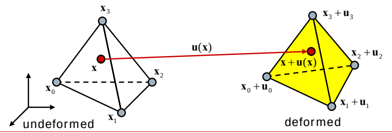
$$\mathbf{u} = b_0\mathbf{u}_0 + b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + b_3\mathbf{u}_3 = \underbrace{\begin{bmatrix}b_0\mathbf{I} & b_1\mathbf{I} & b_2\mathbf{I} & b_3\mathbf{I}\end{bmatrix}}_{\text{3 x 12 matrix}}\begin{bmatrix}\mathbf{u}_0\\\mathbf{u}_1\\\mathbf{u}_2\\\mathbf{u}_3\end{bmatrix}$$

$$\mathbf{b} = \mathbf{X}^{-1}\begin{bmatrix}\mathbf{x}\\1\end{bmatrix}$$

$$\mathbf{u}(\mathbf{x}) = \mathbf{H}(\mathbf{x})\mathbf{u}_e$$

vector of displacements

matrix of **linear** basis functions



Strain Tensor

- ☐ **Linear strain with linear basis results in constant strain**

$$\boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{bmatrix}=\begin{bmatrix}\partial/\partial x_1 & 0 & 0\\ 0 & \partial/\partial x_2 & 0\\ 0 & 0 & \partial/\partial x_3\\ \partial/\partial x_2 & \partial/\partial x_1 & 0\\ 0 & \partial/\partial x_3 & \partial/\partial x_2\\ \partial/\partial x_3 & 0 & \partial/\partial x_1\end{bmatrix}\mathbf{u}(\mathbf{x})=\boxed{\begin{bmatrix}\partial/\partial x_1 & 0 & 0\\ 0 & \partial/\partial x_2 & 0\\ 0 & 0 & \partial/\partial x_3\\ \partial/\partial x_2 & \partial/\partial x_1 & 0\\ 0 & \partial/\partial x_3 & \partial/\partial x_2\\ \partial/\partial x_3 & 0 & \partial/\partial x_1\end{bmatrix}\mathbf{H}(\mathbf{x})}\,\mathbf{u}_e=\mathbf{B}\mathbf{u}_e$$

$$\mathbf{u}(\mathbf{x})=\mathbf{H}(\mathbf{x})\mathbf{u}_e$$

- ☐ $\mathbf{B}\in R^{6\times 12}$ **depends on the original tetrahedron only!**

Stress and Potential Energy

- ☐ **Stress as a function of the displacements**

$$\boldsymbol{\sigma}=\mathbf{E}\boldsymbol{\varepsilon}=\mathbf{E}\mathbf{B}\mathbf{u}_e$$

$$\boldsymbol{\varepsilon}=\mathbf{B}\mathbf{u}_e$$

- ☐ **Energy as a function of the displacements**

$$U_{\mathrm{e}}=\frac{1}{2}\int_{V_e}\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma}\,dV$$

$$=\frac{1}{2}\int_{V_e}(\mathbf{B}\mathbf{u}_e)^{\mathrm{T}}\mathbf{E}\mathbf{B}\mathbf{u}_e\,dV=\frac{1}{2}\mathbf{u}_e^{\mathrm{T}}\left[\int_{V_e}\mathbf{B}^{\mathrm{T}}\mathbf{E}\mathbf{B}\,dV\right]\mathbf{u}_e$$

$$=\frac{1}{2}\mathbf{u}_e^{\mathrm{T}}\left[V_e\mathbf{B}^{\mathrm{T}}\mathbf{E}\mathbf{B}\right]\mathbf{u}_e$$

Stiffness Matrix

- ☐ **Potential energy**

$$U_{\mathrm{e}}=\frac{1}{2}\mathbf{u}_e^{\mathrm{T}}\left[V_e\mathbf{B}^{\mathrm{T}}\mathbf{E}\mathbf{B}\right]\mathbf{u}_e=\frac{1}{2}\mathbf{u}_e^{\mathrm{T}}\mathbf{K}_e\mathbf{u}_e$$

stiffness matrix

- ☐ **Force due to elastic potential energy**

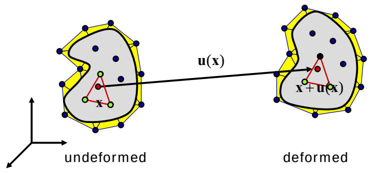
$$\frac{\partial U_{\mathrm{e}}}{\partial \mathbf{u}_e}=\mathbf{K}_e\mathbf{u}_e$$

Finite Element Discretization

- ☐ **For an element**

$$\frac{\partial U(\mathbf{u})}{\partial \mathbf{u}}=\mathbf{f}_{ext}\quad\Rightarrow\quad\frac{\partial U_e}{\partial \mathbf{u}_e}=\mathbf{K}_e\mathbf{u}_e=\mathbf{f}_e$$

governing equations

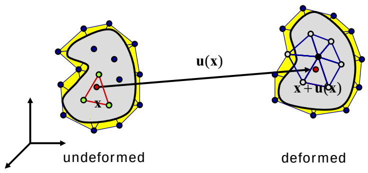


Finite Element Discretization

- ☐ **For an element**

$$\frac{\partial U(\mathbf{u})}{\partial \mathbf{u}}=\mathbf{f}_{ext}\quad\Rightarrow\quad\frac{\partial U_e}{\partial \mathbf{u}_e}=\mathbf{K}_e\mathbf{u}_e=\mathbf{f}_e$$

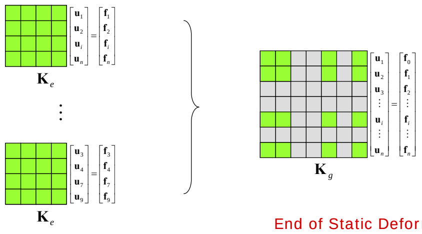
governing equations



Assembling Element Stiffness Matrices

- ☐ **Single element**
- ☐ **Entire body**

$$\mathbf{K}_e\mathbf{u}_e=\mathbf{f}_e\quad\Rightarrow\quad\mathbf{K}_g\mathbf{u}_g=\mathbf{f}_g$$



$$\mathbf{K}_e\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\\ \mathbf{u}_i\\ \mathbf{u}_n\end{bmatrix}=\begin{bmatrix}\mathbf{f}_1\\ \mathbf{f}_2\\ \mathbf{f}_i\\ \mathbf{f}_n\end{bmatrix}\qquad\vdots\qquad\mathbf{K}_e\begin{bmatrix}\mathbf{u}_3\\ \mathbf{u}_4\\ \mathbf{u}_7\\ \mathbf{u}_9\end{bmatrix}=\begin{bmatrix}\mathbf{f}_3\\ \mathbf{f}_4\\ \mathbf{f}_7\\ \mathbf{f}_9\end{bmatrix}$$

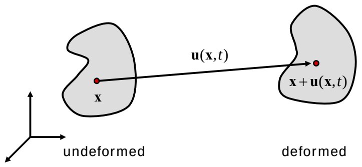
$$\mathbf{K}_g\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\\ \mathbf{u}_3\\ \vdots\\ \mathbf{u}_i\\ \vdots\\ \mathbf{u}_n\end{bmatrix}=\begin{bmatrix}\mathbf{f}_0\\ \mathbf{f}_1\\ \mathbf{f}_2\\ \vdots\\ \mathbf{f}_i\\ \vdots\\ \mathbf{f}_n\end{bmatrix}$$

End of Static Deformation

Dynamic Deformation

- ☐ **Governing equations: Euler-Lagrange equations**

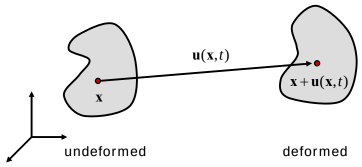
$$\underbrace{\frac{d}{dt}\left(\frac{\partial T(\mathbf{u})}{\partial \dot{\mathbf{u}}}\right)}_{\text{kinetic energy}} + \underbrace{\frac{\partial U(\mathbf{u})}{\partial \mathbf{u}}}_{\text{elastic potential energy}} = \mathbf{f}_{ext} \longleftarrow \text{external force}$$



Mass Matrix

- ☐ **Kinetic energy**

$$T_e = \frac{1}{2}\int_{V_e} \rho(\mathbf{x})\frac{d}{dt}[\mathbf{x}+\mathbf{u}(\mathbf{x},t)]\cdot\frac{d}{dt}[\mathbf{x}+\mathbf{u}(\mathbf{x},t)]dV$$



Mass Matrix

- ☐ **Kinetic energy**

$$\begin{aligned}T_e &= \frac{1}{2}\int_{V_e} \rho(\mathbf{x})\frac{d}{dt}[\mathbf{x}+\mathbf{u}(\mathbf{x},t)]\cdot\frac{d}{dt}[\mathbf{x}+\mathbf{u}(\mathbf{x},t)]dV = \frac{1}{2}\int_{V_e}\rho(\mathbf{x})\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\,dV \\ &= \frac{1}{2}\int_{V_e}\rho(\mathbf{x})[\mathbf{H}(\mathbf{x})\dot{\mathbf{u}}_e]^{\mathrm{T}}[\mathbf{H}(\mathbf{x})\dot{\mathbf{u}}_e]dV \\ &= \frac{1}{2}\dot{\mathbf{u}}_e^{\mathrm{T}}\left[\int_{V_e}\rho(\mathbf{x})\mathbf{H}(\mathbf{x})^{\mathrm{T}}\mathbf{H}(\mathbf{x})\,dV\right]\dot{\mathbf{u}}_e = \frac{1}{2}\dot{\mathbf{u}}_e^{\mathrm{T}}\underbrace{\mathbf{M}_e}_{\text{mass matrix}}\dot{\mathbf{u}}_e\end{aligned}$$

$$\mathbf{u}(\mathbf{x}) = \mathbf{H}(\mathbf{x})\mathbf{u}_e$$

$$\dot{\mathbf{u}}(\mathbf{x}) = \mathbf{H}(\mathbf{x})\dot{\mathbf{u}}_e$$

- ☐ **Force due to kinetic energy**

$$\frac{d}{dt}\left(\frac{\partial T_e}{\partial \dot{\mathbf{u}}_e}\right) = \frac{d}{dt}(\mathbf{M}_e\dot{\mathbf{u}}_e) = \mathbf{M}_e\ddot{\mathbf{u}}_e$$

Damping Matrix

- ☐ **Governing equations: Euler-Lagrange equations**

$$\frac{d}{dt}\left(\frac{\partial T(\mathbf{u})}{\partial \dot{\mathbf{u}}}\right) + \frac{\partial U(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{f}_{ext}$$

$$\Downarrow$$

$$\mathbf{M}_e\ddot{\mathbf{u}}_e + \mathbf{K}_e\mathbf{u}_e = \mathbf{f}_e$$

Damping Matrix

- ☐ **Governing equations with damping terms**

$$\frac{d}{dt}\left(\frac{\partial T(\mathbf{u})}{\partial \dot{\mathbf{u}}}\right) + \mathbf{C}\dot{\mathbf{u}} + \frac{\partial U(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{f}_{ext}$$

$$\Downarrow$$

$$\mathbf{M}_e\ddot{\mathbf{u}}_e + \underbrace{\mathbf{C}_e}_{\text{damping matrix}}\dot{\mathbf{u}}_e + \mathbf{K}_e\mathbf{u}_e = \mathbf{f}_e$$

- Adopting proportional (Rayleigh) damping: $\mathbf{C}_e = \alpha\mathbf{M}_e + \beta\mathbf{K}_e$
- Or, incorporating visco-elastic formulation

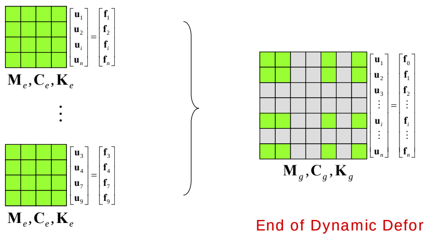
Assembling Element Matrices

- ☐ **Single element**

$$\mathbf{M}_e\ddot{\mathbf{u}}_e + \mathbf{C}_e\dot{\mathbf{u}}_e + \mathbf{K}_e\mathbf{u}_e = \mathbf{f}_e$$

- ☐ **Entire body**

$$\mathbf{M}_g\ddot{\mathbf{u}}_g + \mathbf{C}_g\dot{\mathbf{u}}_g + \mathbf{K}_g\mathbf{u}_g = \mathbf{f}_g$$



End of Dynamic Deformation